PHY456H1F: Quantum Mechanics II. Lecture 15 (Taught by Prof J.E. Sipe). Rotation operator in spin space

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Contents

1. Disclaimer.

Peeter's lecture notes from class. May not be entirely coherent.

2. Rotation operator in spin space.

We can formally expand our rotation operator in Taylor series

$$e^{-i\theta \hat{\mathbf{n}} \cdot \mathbf{S}/\hbar} = I + (-i\theta \hat{\mathbf{n}} \cdot \mathbf{S}/\hbar) + \frac{1}{2!} (-i\theta \hat{\mathbf{n}} \cdot \mathbf{S}/\hbar)^2 + \frac{1}{3!} (-i\theta \hat{\mathbf{n}} \cdot \mathbf{S}/\hbar)^3 + \cdots \tag{1}$$

or

$$\begin{aligned}
e^{-i\theta \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}/2} &= I + (-i\theta \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}/2) + \frac{1}{2!} (-i\theta \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}/2)^2 + \frac{1}{3!} (-i\theta \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}/2)^3 + \cdots \\
&= \sigma_0 + \left(\frac{-i\theta}{2}\right) (\hat{\mathbf{n}} \cdot \boldsymbol{\sigma}) + \frac{1}{2!} \left(\frac{-i\theta}{2}\right) (\hat{\mathbf{n}} \cdot \boldsymbol{\sigma})^2 + \frac{1}{3!} \left(\frac{-i\theta}{2}\right) (\hat{\mathbf{n}} \cdot \boldsymbol{\sigma})^3 + \cdots \\
&= \sigma_0 + \left(\frac{-i\theta}{2}\right) (\hat{\mathbf{n}} \cdot \boldsymbol{\sigma}) + \frac{1}{2!} \left(\frac{-i\theta}{2}\right) \sigma_0 + \frac{1}{3!} \left(\frac{-i\theta}{2}\right) (\hat{\mathbf{n}} \cdot \boldsymbol{\sigma}) + \cdots \\
&= \sigma_0 \left(1 - \frac{1}{2!} \left(\frac{\theta}{2}\right)^2 + \cdots \right) + (\hat{\mathbf{n}} \cdot \boldsymbol{\sigma}) \left(\frac{\theta}{2} - \frac{1}{3!} \left(\frac{\theta}{2}\right)^3 + \cdots \right) \\
&= \cos(\theta/2)\sigma_0 + \sin(\theta/2)(\hat{\mathbf{n}} \cdot \boldsymbol{\sigma})
\end{aligned}$$

where we've used the fact that $(\hat{\mathbf{n}} \cdot \boldsymbol{\sigma})^2 = \sigma_0$.
So our representation of the spin operator is

$$
\begin{aligned}
e^{-i\theta \hat{\mathbf{n}}\cdot \mathbf{S}/\hbar} &\rightarrow \cos(\theta/2)\sigma_0 + \sin(\theta/2)(\hat{\mathbf{n}}\cdot \boldsymbol{\sigma}) \\
&= \cos(\theta/2)\sigma_0 + \sin(\theta/2)\left(n_x \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + n_y \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + n_z \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) \\
&= \begin{bmatrix} \cos(\theta/2) - i n_z \sin(\theta/2) & -i(n_x - i n_y)\sin(\theta/2) \\ -i(n_x + i n_y)\sin(\theta/2) & \cos(\theta/2) + i n_z \sin(\theta/2) \end{bmatrix}
\end{aligned}
\tag{2}
$$

Note that, in particular,

$$
e^{-2\pi i \hat{\mathbf{n}}\cdot \mathbf{S}/\hbar} \rightarrow \cos \pi \sigma_0 = -\sigma_0 \tag{3}
$$

This "rotates" the ket, but introduces a phase factor.

Can do this in general for other degrees of spin, for $s = 1/2, 3/2, 5/2, \cdots$.

2.1. *Unfortunate interjection by me*

I mentioned the half angle rotation operator that requires a half angle operator sandwich. Prof. Sipe thought I might be talking about a Heisenberg picture representation, where we have something like this in expectation values

$$
\left|\psi'\right\rangle = e^{-i\theta \hat{\mathbf{n}}\cdot \mathbf{J}/\hbar} \left|\psi\right\rangle \tag{4}
$$

so that

$$
\left\langle \psi' \right| \mathcal{O} \left| \psi' \right\rangle = \left\langle \psi \right| e^{i\theta \hat{\mathbf{n}}\cdot \mathbf{J}/\hbar} \mathcal{O} e^{-i\theta \hat{\mathbf{n}}\cdot \mathbf{J}/\hbar} \left|\psi\right\rangle \tag{5}
$$

However, what I was referring to, was that a general rotation of a vector in a Pauli matrix basis

$$
R(\sum a_k \sigma_k) = R(\mathbf{a}\cdot \boldsymbol{\sigma}) \tag{6}
$$

can be expressed by sandwiching the Pauli vector representation by two half angle rotation operators like our spin 1/2 operators from class today

$$
R(\mathbf{a}\cdot \boldsymbol{\sigma}) = e^{-\theta \hat{\mathbf{u}}\cdot\sigma \hat{\mathbf{v}}\cdot\sigma/2} \mathbf{a}\cdot \boldsymbol{\sigma} e^{\theta \hat{\mathbf{u}}\cdot\sigma \hat{\mathbf{v}}\cdot\sigma/2} \tag{7}
$$

where $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ are two non-colinear orthogonal unit vectors that define the oriented plane that we are rotating in.

For example, rotating in the $x-y$ plane, with $\hat{\mathbf{u}} = \hat{\mathbf{x}}$ and $\hat{\mathbf{v}} = \hat{\mathbf{y}}$, we have

$$
R(\mathbf{a}\cdot \boldsymbol{\sigma}) = e^{-\theta\sigma_1\sigma_2/2}(a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3) e^{\theta\sigma_1\sigma_2/2} \tag{8}
$$

Observe that these exponentials commute with σ_3, leaving

$$
\begin{aligned}
R(\mathbf{a}\cdot \boldsymbol{\sigma}) &= (a_1\sigma_1 + a_2\sigma_2) e^{\theta\sigma_1\sigma_2} + a_3\sigma_3 \\
&= (a_1\sigma_1 + a_2\sigma_2)(\cos\theta + \sigma_1\sigma_2 \sin\theta) + a_3\sigma_3 \\
&= \sigma_1(a_1\cos\theta - a_2\sin\theta) + \sigma_2(a_2\cos\theta + a_1\sin\theta) + \sigma_3(a_3)
\end{aligned}
$$

yielding our usual coordinate rotation matrix. Expressed in terms of a unit normal to that plane, we form the normal by multiplication with the unit spatial volume element $I = \sigma_1\sigma_2\sigma_3$. For example:

$$\sigma_1\sigma_2\sigma_3(\sigma_3) = \sigma_1\sigma_2 \tag{9}$$

and can in general write a spatial rotation in a Pauli basis representation as a sandwich of half angle rotation matrix exponentials

$$R(\mathbf{a}\cdot\boldsymbol{\sigma}) = e^{-I\theta(\hat{\mathbf{n}}\cdot\boldsymbol{\sigma})/2}(\mathbf{a}\cdot\boldsymbol{\sigma})e^{I\theta(\hat{\mathbf{n}}\cdot\boldsymbol{\sigma})/2} \tag{10}$$

when $\hat{\mathbf{n}}\cdot\mathbf{a} = 0$ we get the complex-number like single sided exponential rotation exponentials (since $\mathbf{a}\cdot\boldsymbol{\sigma}$ commutes with $\mathbf{n}\cdot\boldsymbol{\sigma}$ in that case)

$$R(\mathbf{a}\cdot\boldsymbol{\sigma}) = (\mathbf{a}\cdot\boldsymbol{\sigma})e^{I\theta(\hat{\mathbf{n}}\cdot\boldsymbol{\sigma})} \tag{11}$$

I believe it was pointed out in one of [1] or [2] that rotations expressed in terms of half angle Pauli matrices has caused some confusion to students of quantum mechanics, because this 2π "rotation" only generates half of the full spatial rotation. It was argued that this sort of confusion can be avoided if one observes that these half angle rotations exponentials are exactly what we require for general spatial rotations, and that a pair of half angle operators are required to produce a full spatial rotation.

The book [1] takes this a lot further, and produces a formulation of spin operators that is devoid of the normal scalar imaginary i (using the Clifford algebra spatial unit volume element instead), and also does not assume a specific matrix representation of the spin operators. They argue that this leads to some subtleties associated with interpretation, but at the time I was attempting to read that text I did know enough QM to appreciate what they were doing, and haven't had time to attempt a new study of that content.

3. Spin dynamics

At least classically, the angular momentum of charged objects is associated with a magnetic moment as illustrated in figure (1)

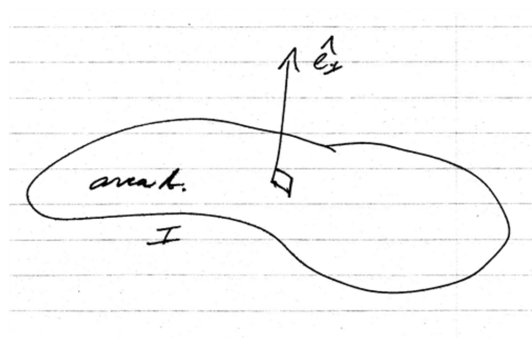

Figure 1: Magnetic moment due to steady state current

$$\boldsymbol{\mu} = IA\mathbf{e}_\perp \tag{12}$$

In our scheme, following the (cgs?) text conventions of [3], where the $\mathbf{E}$ and $\mathbf{B}$ have the same units, we write

$$\boldsymbol{\mu} = \frac{IA}{c}\mathbf{e}_\perp \tag{13}$$

For a charge moving in a circle as in figure (2)

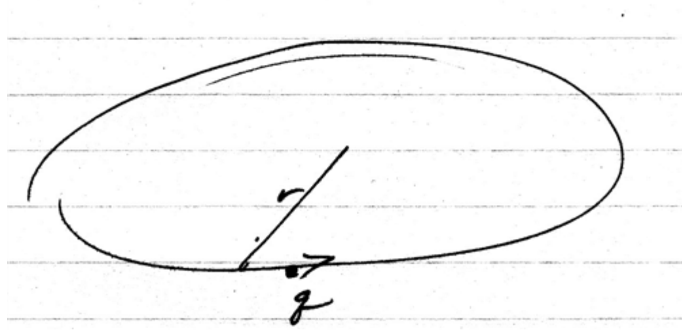

Figure 2: Charge moving in circle.

$$\begin{aligned} I &= \frac{\text{charge}}{\text{time}} \\ &= \frac{\text{distance}}{\text{time}}\frac{\text{charge}}{\text{distance}} \\ &= \frac{qv}{2\pi r} \end{aligned} \tag{14}$$

so the magnetic moment is

$$\begin{aligned} \mu &= \frac{qv}{2\pi r}\frac{\pi r^2}{c} \\ &= \frac{q}{2mc}(mvr) \\ &= \gamma L \end{aligned} \tag{15}$$

Here γ is the gyromagnetic ratio

Recall that we have a torque, as shown in figure (3)

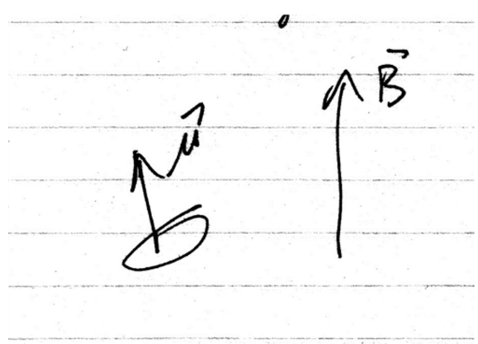

Figure 3: Induced torque in the presence of a magnetic field.

$$\mathbf{T} = \boldsymbol{\mu} \times \mathbf{B} \tag{16}$$

tending to line up $\boldsymbol{\mu}$ with $\mathbf{B}$. The energy is then

$$-\boldsymbol{\mu} \cdot \mathbf{B} \tag{17}$$

Also recall that this torque leads to precession as shown in figure (4)

$$\frac{d\mathbf{L}}{dt} = \mathbf{T} = \gamma \mathbf{L} \times \mathbf{B}, \tag{18}$$

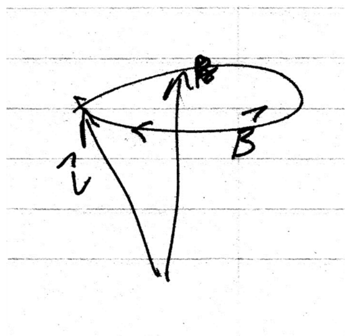

Figure 4: Precession due to torque.

with precession frequency

$$\boldsymbol{\omega} = -\gamma \mathbf{B}. \tag{19}$$

For a current due to a moving electron

$$\gamma = -\frac{e}{2mc} < 0 \tag{20}$$

where we are, here, writing for charge on the electron $-e$.

Question: steady state currents only? . Yes, this is only true for steady state currents.

For the translational motion of an electron, even if it is not moving in a steady way, regardless of it's dynamics

$$\boldsymbol{\mu}_0 = -\frac{e}{2mc} \mathbf{L} \tag{21}$$

Now, back to quantum mechanics, we turn $\boldsymbol{\mu}_0$ into a dipole moment operator and $\mathbf{L}$ is "promoted" to an angular momentum operator.

$$H_{\text{int}} = -\boldsymbol{\mu}_0 \cdot \mathbf{B} \tag{22}$$

What about the "spin"?
Perhaps

$$\boldsymbol{\mu}_s = \gamma_s \mathbf{S} \tag{23}$$

we write this as

$$\boldsymbol{\mu}_s = g \left(-\frac{e}{2mc} \right) \mathbf{S} \tag{24}$$

so that

$$\gamma_s = -\frac{ge}{2mc} \tag{25}$$

Experimentally, one finds to very good approximation

$$g = 2 \tag{26}$$

There was a lot of trouble with this in early quantum mechanics where people got things wrong, and canceled the wrong factors of 2.

In fact, Dirac's relativistic theory for the electron predicts $g = 2$.

When this is measured experimentally, one does not get exactly $g = 2$, and a theory that also incorporates photon creation and destruction and the interaction with the electron with such (virtual) photons. We get

$$\begin{aligned} g_{\text{theory}} &= 2\,(1.001159652140(\pm 28)) \\ g_{\text{experimental}} &= 2\,(1.0011596521884(\pm 43)) \end{aligned} \tag{27}$$

Richard Feynman compared the precision of quantum mechanics, referring to this measurement, "to predicting a distance as great as the width of North America to an accuracy of one human hair's breadth".

References

[1] C. Doran and A.N. Lasenby. *Geometric algebra for physicists*. Cambridge University Press New York, Cambridge, UK, 1st edition, 2003. 2.1

[2] D. Hestenes. *New Foundations for Classical Mechanics*. Kluwer Academic Publishers, 1999. 2.1

[3] BR Desai. *Quantum mechanics with basic field theory*. Cambridge University Press, 2009. 3